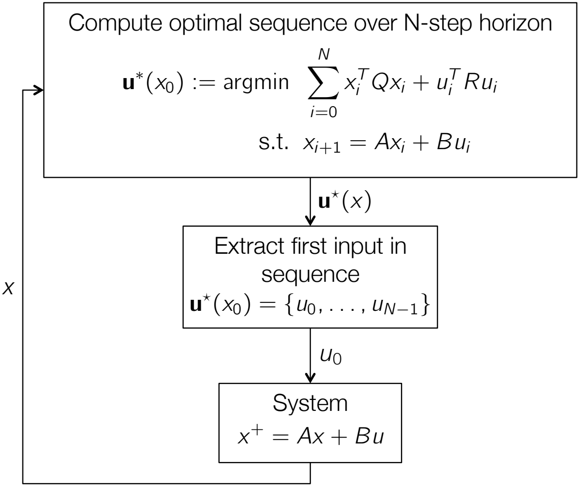
Model Predictive Control

Lecture: Introduction to Convex Optimization

Colin Jones

Laboratoire d'Automatique, EPFL

Recall: Receding Horizon Control



For unconstrained systems, this is a **constant linear controller**
However, can extend this concept to much more complex systems (MPC)

What's the Prediction For?

Horizon provides a tradeoff between short-term and long-term benefits.

Infinite-horizon:

- Cost is finite only if the system is stable
- Minimizing infinite-horizon prediction stabilizes the system

Cannot consider infinite-horizons when solving an optimization problem

- We will 'fake' infinite-horizon when solving MPC problems

Linear Quadratic Regulator

$$V^\star(x_0) := \min_{\mathbf{u}} \quad \sum_{k=0}^{\infty} {x_k}^T Q x_k + {u_k}^T R u_k \quad \text{s.t. } x_{k+1} = Ax_k + Bu_k$$

Can solve the infinite-horizon predictive control problem in closed-form:

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA$$

The optimal input is the constant state feedback

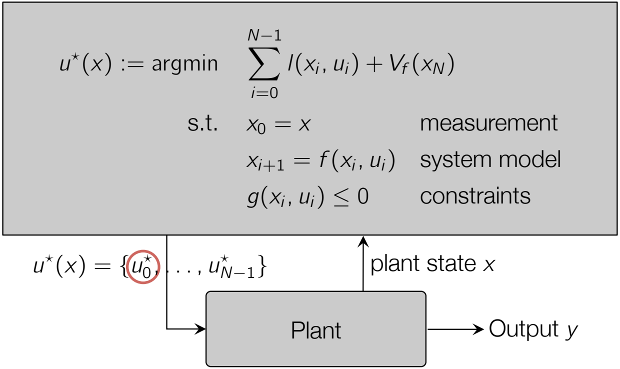
$$u = Kx \qquad\qquad K = -(R + B^T PB)^{-1} B^T PA$$

The optimal cost function $V^\star(x) = x^T Px$ is a Lyapunov function for the closed-loop system $x^+ = (A + BK)x$.

Outline

1. Optimization in MPC
2. Main Concepts
3. Convex Optimization
4. Linear and Quadratic Programming
5. Constrained Minimization: Interior-point Methods
   - Concept
   - Unconstrained Minimization
   - Barrier Interior-Point Method
6. Summary of Exercise Session

MPC: Optimization in the loop



At each sample time:

- Measure /estimate current state
- **Find the optimal input sequence for the entire planning window N**
- Implement only the first control action

Optimization Problems arising in MPC

Linear Systems

- Linear system dynamics
- Continuous set of states and inputs, e.g., $x \in [x_{\min}, x_{\max}]$, $u \in [u_{\min}, u_{\max}]$
- Example: Chemical processes

Nonlinear Systems

- Nonlinear system dynamics
- Continuous set of states and inputs, e.g., $x \in [x_{\min}, x_{\max}]$, $u \in [u_{\min}, u_{\max}]$
- Example: Kites

Hybrid Systems

- Mixed dynamics that are both continuous and discrete, e.g.
  $$\begin{cases} x_{k+1} = -c_1 & x_k \geq x_{\max} \\ x_{k+1} = c_2 - c_1 & x_k < x_{\max} \end{cases}$$
- Continuous set of states and inputs
- Example: Walking robot

Discrete Decision Variables

- Inputs and/or states can only take discrete values, e.g. $u \in \{1, 2, 3, 4, 5\}$
- Example: Internet

Outline

1. Optimization in MPC

2. Main Concepts

3. Convex Optimization

4. Linear and Quadratic Programming

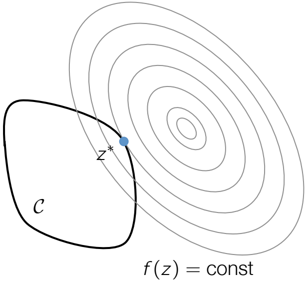
5. Constrained Minimization: Interior-point Methods
    - Concept
    - Unconstrained Minimization
    - Barrier Interior-Point Method

6. Summary of Exercise Session

Mathematical Optimization

Mathematical optimization problem is generally formulated as

$$\begin{aligned} \text{minimize } & f(z) \\ \text{s.t. } & g_i(z) \leq 0, \ i = 1, \dots, m \\ & h_i(z) = 0, \ i = 1, \dots, p \end{aligned}$$



- $z = [z_1, \dots, z_n]$: optimization variables
- $f : \mathbb{R}^n \to \mathbb{R}$: objective or cost function
- $g : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$: inequality constraint functions
- $h : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$: equality constraint functions
- z is **feasible** or admissible if it satisfies the constraints
- $\mathcal{C} := \{z \mid g_i(z) \leq 0, \ i = 1, \dots, m, h_i(z) = 0, \ i = 1, \dots, p\}$: set of feasible or admissible decisions, or **feasible set**

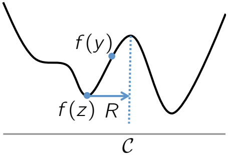
Optimality

Optimal value: smallest possible cost
$p^* \triangleq \inf \{ f(z) \mid g_i(z) \leq 0 \; i = 1, \ldots, m, h_i(z) = 0, \; i = 1, \ldots, p \}$

Optimizer: feasible z that achieves smallest cost p^*, i.e., $z^* \in \mathcal{C}$ with $p^* = f(z^*)$; set of all optimizers is denoted by Z_{opt} (optimizer is not always unique).

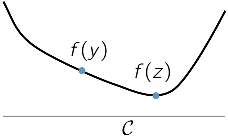
- $z \in \mathcal{C}$ is **locally optimal** if, for some $R > 0$, it satisfies

$$y \in \mathcal{C}, \|y - z\| \leq R \Rightarrow f(y) \geq f(z)$$



- $z \in \mathcal{C}$ is **globally optimal** if it satisfies

$$y \in \mathcal{C} \Rightarrow f(y) \geq f(z)$$



- If $p^* = -\infty$ the problem is **unbounded below**
- If $\mathcal{C}$ is empty, then the problem is said to be **infeasible** (convention: $p^* = \infty$)
- If $m = p = 0$ the problem is said to be **unconstrained**

Solving nonlinear optimization problems

Traditional techniques for general nonconvex problems involve compromises, e.g., very long computation time, or not always finding the solution:

Local optimization methods
Find a point that minimizes f among feasible points near it

- Fast, can handle large problems
- Requires initial guess
- Provides no information about distance to (global) optimum

Global optimization methods
Find the (global) solution

- Worst-case complexity grows exponentially with problem size

Exceptions
Certain problem classes can be solved efficiently and reliably:
e.g. **convex optimization problems**

Outline

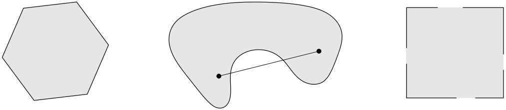
1. Optimization in MPC
2. Main Concepts
3. Convex Optimization
4. Linear and Quadratic Programming
5. Constrained Minimization: Interior-point Methods
   - Concept
   - Unconstrained Minimization
   - Barrier Interior-Point Method
6. Summary of Exercise Session

Convex Sets

Convex set: A set $S \in \mathbb{R}^s$ is **convex** if

$$\lambda z_1 + (1-\lambda) z_2 \in S \text{ for all } z_1, z_2 \in S, \lambda \in [0,1]$$

i.e convex set contains line segment between any two points in the set
Examples: one convex, two non-convex sets

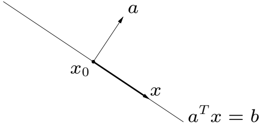


Convex combination of $z_1, \ldots, z_k$: Any point z of the form

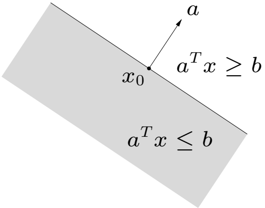
$$z = \theta_1 z_1 + \theta_2 z_2 + \ldots + \theta_k z_k \text{ with } \theta_1 + \ldots + \theta_k = 1, \theta_i \geq 0$$

Convex sets: Hyperplanes and Halfspaces

- **Hyperplane:** Set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$



- **Halfspace:** Set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



- Useful representation: $\left\{x \;\middle|\; a^T(x - x_0) \leq 0\right\}$
  a is normal vector, x_0 lies on the boundary
- Hyperplanes are affine and convex, halfspaces are convex

Convex sets: Polyhedra

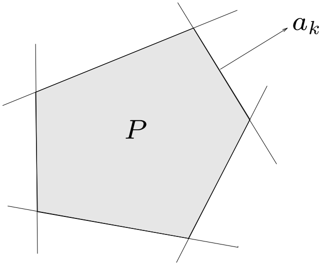
Polyhedron

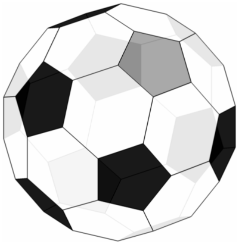
A **polyhedron** is the intersection of a finite number of halfspaces.

$$P := \left\{x \mid a_i^T x \le b_i,\ i = 1, \dots, n\right\}$$

A **polytope** is a bounded polyhedron.

Often written as $P := \{x \mid Ax \le b\}$, for matrix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, where the inequality is understood row-wise.

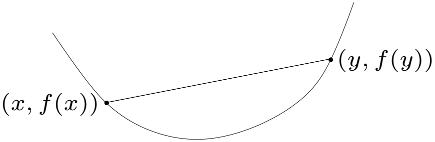




Convex function: Definition

- **Convex function:**
  A function $f : S \to \mathbb{R}$ is convex if S is convex and

$$f(\lambda z_1 + (1-\lambda) z_2) \le \lambda f(z_1) + (1-\lambda) f(z_2)$$
$$\text{for all } z_1, z_2 \in S, \lambda \in [0,1]$$



- A function $f : S \to \mathbb{R}$ is **strictly convex** if S is convex and

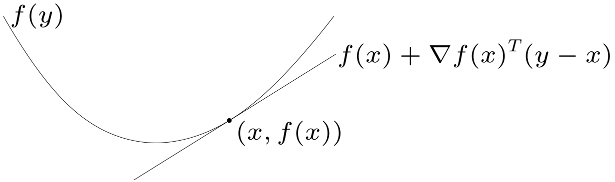
$$f(\lambda z_1 + (1-\lambda) z_2) < \lambda f(z_1) + (1-\lambda) f(z_2)$$
$$\text{for all } z_1, z_2 \in S, \lambda \in (0,1)$$

- A function $f : S \to \mathbb{R}$ is **concave** if S is convex and $-f$ is convex.

First and second order condition for convexity

First-order condition: Differentiable f with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \textbf{dom}\, f$$



$\rightarrow$ First-order approximation of f is global underestimator

Second-order condition: Twice differentiable f with convex domain convex iff

$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \textbf{dom}\, f$$

Convex functions–Examples

Examples on $\mathbb{R}$:

- Exponential: e^{ax}, for any $a \in \mathbb{R}$
- Powers: x^a on $\mathbb{R}_+$ for $a \geq 1$ or $a \leq 0$ (otherwise concave)
- Logarithm: $-\log x$ on $\mathbb{R}_+$

Examples on $\mathbb{R}^n$:

- Affine function: $f(x) = a^T x + b$
- Norms: $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

Convex optimization problem

Convex optimization problem in standard form

$$\begin{aligned} \min\ & f(z) \\ \text{s.t. } & g_i(z) \leq 0,\ i = 1, \ldots, m \\ & c_i^T z = b_i,\ i = 1, \ldots, p \end{aligned}$$

- $f, g_1, \ldots, g_m$ are convex
- equality constraints are affine

often rewritten as

$$\begin{aligned} \min\ & f(z) \\ \text{s.t. } & g(z) \leq 0 \\ & Cz = b \end{aligned}$$

where $C \in \mathbb{R}^{p \times n}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$.

Important property: Feasible set of a convex optimization problem is convex.

Local and global optimality in convex optimization

Lemma: Convex problems: Local optima are global optima

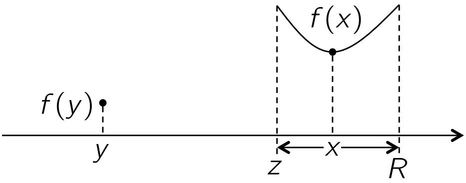
Any locally optimal point of a convex problem is globally optimal.

Proof:

Assume x locally optimal and a feasible y such $f(y) < f(x)$.

x locally optimal implies that there exists an $R > 0$ such that

$$\|z - x\|_2 \leq R \ \Rightarrow \ f(z) \geq f(x)$$



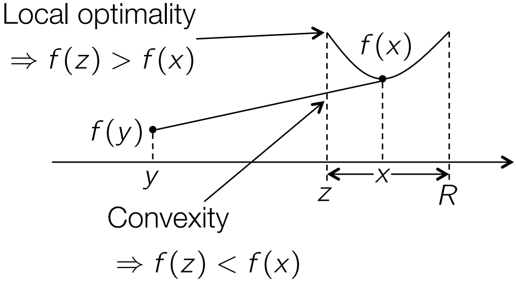
Local and global optimality in convex optimization

Lemma: Convex problems: Local optima are global optima
Any locally optimal point of a convex problem is globally optimal.

Proof:
Assume x locally optimal and a feasible y such $f(y) < f(x)$.

x locally optimal implies that there exists an $R > 0$ such that

$$\|z - x\|_2 \leq R \;\Rightarrow\; f(z) \geq f(x)$$



Recap: Convex optimization

- Convex optimization problem:
  - Convex cost function
  - Convex inequality constraints
  - Affine equality constraints
- Benefit of convex problems: Local = Global optimality
- Only need to find one minimum, it is the global minimum!
- Convex optimization problems can be solved efficiently

Outline

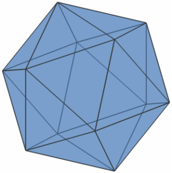
1. Optimization in MPC
2. Main Concepts
3. Convex Optimization
4. Linear and Quadratic Programming
5. Constrained Minimization: Interior-point Methods
   - Concept
   - Unconstrained Minimization
   - Barrier Interior-Point Method
6. Summary of Exercise Session

Linear Program (LP)

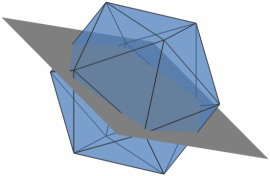
$$\begin{aligned} \min\ & c^T z \\ \text{s.t. } & Gz \leq d \\ & Cz = b \end{aligned}$$

where $z \in \mathbb{R}^n$.

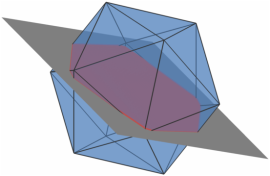
- Convex optimization problem with affine objective and constraint functions
- Feasible set P is a polyhedron



(a) $Gz \leq d$



(b) $C_i^T z = b_i$



(c) $Gz \leq d \cap C_i^T z = b_i$

- If P is empty, then the problem is infeasible

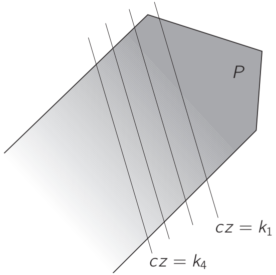
Graphical Interpretation and Solutions Properties

Denote by p^* the optimal value and by Z_{opt} the set of optimizers

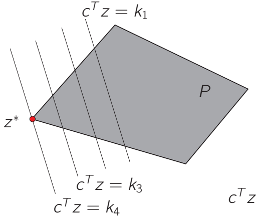
Case 1. The LP solution is unbounded, i.e., $p^* = -\infty$.

Case 2. The LP solution is bounded, i.e., $p^* > -\infty$ and the optimizer is unique. Z_{opt} is a singleton.

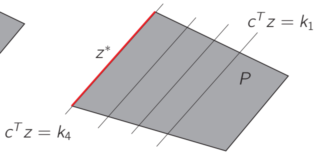
Case 3. The LP solution is bounded and there are multiple optima. Z_{opt} is a subset of $\mathbb{R}^s$, which can be bounded or unbounded.



(a) Case 1



(b) Case 2



(c) Case 3

Quadratic program (QP)

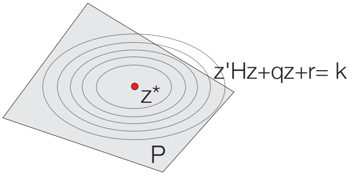
$$
\begin{aligned}
\min \ & \frac{1}{2} z^T H z + q^T z + r \\
\text{s.t. } & Gz \leq d \\
& Cz = b
\end{aligned}
$$

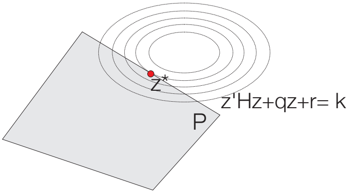
where $z \in \mathbb{R}^n$, $H \in \mathbb{R}^{n \times n}$.

- Convex if $H \succeq 0$ (hard problem if $H \not\succeq 0$)
- Let P be the feasible set.

Two cases can occur if P is not empty:

Case 1. The optimizer lies strictly inside the feasible polyhedron
Case 2. The optimizer lies on the boundary of the feasible polyhedron





Standard optimization problems in MPC

Most common MPC problems based on

- Linear system model
- Linear constraints
- Linear norm or quadratic cost

$\rightarrow$ Result in linear or quadratic programs

Linear norm vs. quadratic cost:

Linear norm $\rightarrow$ LP:

- Very easy to solve
- Possibly non-unique solutions
- Minimize 'quantity' of something
- Far away from origin: slow action
- Close to the origin: a lot of action, jumping, dead-beat and nervous behavior

Quadratic $\rightarrow$ QP:

- More comp. effort (still easy)
- Unique solution
- Energy arguments
- Relation to LQ control
- Far away from origin: a lot of action
- Close to the origin: smooth action

Outline

1. Optimization in MPC
2. Main Concepts
3. Convex Optimization
4. Linear and Quadratic Programming
5. Constrained Minimization: Interior-point Methods
    - Concept
    - Unconstrained Minimization
    - Barrier Interior-Point Method
6. Summary of Exercise Session

Constrained Minimization Problem

Consider the following problem with inequality constraints

$$\begin{aligned} &\min\ f(z) \\ &\text{s.t. } g_i(z) \le 0,\ i = 1, \ldots, m \end{aligned}$$

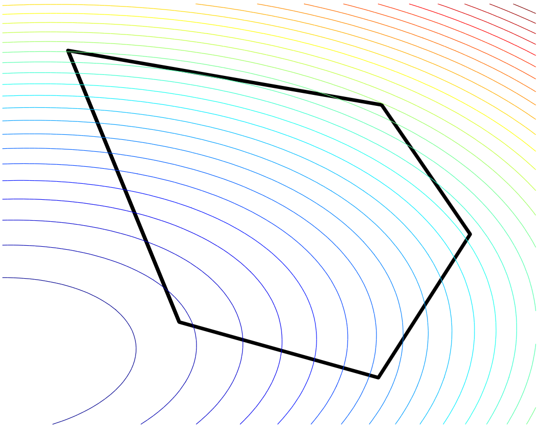
- f, g_i convex, twice continuously differentiable
- We assume p^* is finite and attained
- We assume problem is strictly feasible: there exists a $\tilde{z}$ with

$$\tilde{z} \in \text{ domain of } f, \quad g_i(\tilde{z}) < 0, i = 1, \ldots, m$$

Idea: There exist many methods for unconstrained minimization

$\Rightarrow$ Reformulate problem as an unconstrained problem

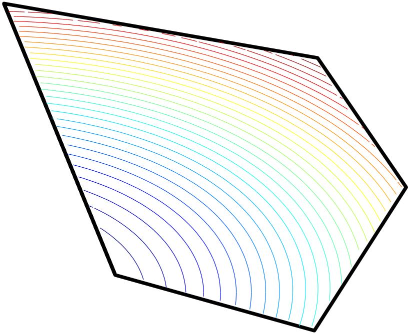
Graphical Illustration



Optimize a function over a set

Graphical Illustration

Define function as ∞ if constraints violated.



Optimize a function over $\mathbb{R}^n$

Barrier method

$$\min_z f(z) + \kappa\phi(z)$$

Reformulate via indicator function:

$$\phi(z) = \sum_{i=1}^{m} I_-(g_i(z)), \quad \kappa = 1$$

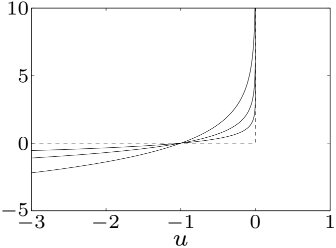
where $I_-(u) = 0$ if $u \leq 0$ and $I_- = \infty$ otherwise (indicator function of $\mathbb{R}_-$)

- Augmented cost is not differentiable

Approximation via logarithmic barrier:

$$\phi(z) = -\sum_{i=1}^{m} log(-g_i(z))$$

- For $\kappa > 0$ smooth approximation of indicator function
- Approximation improves as $\kappa \to 0$



Logarithmic Barrier Function

$$\phi(z) = -\sum_{i=1}^{m} log(-g_i(z)), \qquad \text{domain } \phi = \{z \mid g_1(z) \leq 0, \ldots, g_m(z) \leq 0\}$$

- Convex, smooth on its domain
- $\phi(z) \to \infty$ as z approaches boundary of domain and of the inequality constraints
- $argmin_z \phi(z)$ is called **analytic center** of inequalities $g_1 < 0, \ldots, g_m < 0$
- Twice continously differentiable with derivatives

$$\nabla \phi(z) = \sum_{i=1}^{m} \frac{1}{-g_i(z)} \nabla g_i(z)$$

$$\nabla^2 \phi(z) = \sum_{i=1}^{m} \frac{1}{g_i(z)^2} \nabla g_i(z) \nabla g_i(z)^T + \frac{1}{-g_i(z)} \nabla^2 g_i(z)$$

Central Path

- Define $z^*(\kappa)$ as the solution of

$$\min_z f(z) + \kappa\phi(z)$$

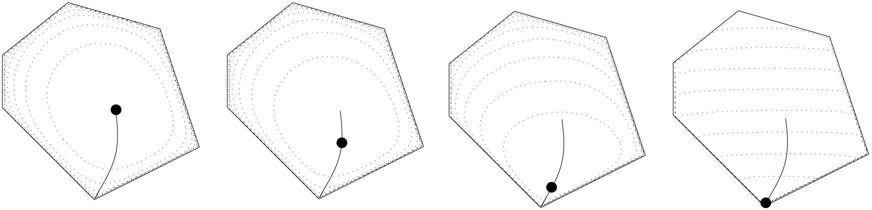
  (assume minimizer exists and is unique for each $\kappa > 0$)
- Barrier parameter κ determines relative weight of objective and barrier
- Barrier 'traps' $z(\kappa)$ in strictly feasible set
- **Central path** is defined as $\{z^*(\kappa) \mid \kappa > 0\}$
- For given κ can compute $z^*(\kappa)$ by solving smooth unconstrained minimization problem
- Intuitively $z^*(\kappa)$ converges to optimal solution as $\kappa \to 0$

Example: Central path for an LP

$$\min\ c^T z$$
$$\text{s.t. } a_i^T x \le b_i, i = 1, \dots, 6$$

$x \in \mathbb{R}^2$, c points up

$\kappa = 1000$ $\kappa = 1$ $\kappa = 1/5$ $\kappa = 1/100$



Path-following Method

Idea: Follow central path to the optimal solution

Solve sequence of smooth unconstrained problems:

$$z^*(\kappa) = \text{argmin}_z\, f(z) + \kappa\phi(z)$$

- Assume current solution is on the central path $z^{(k)} = z^*(\kappa^{(k)})$
- Update $\kappa^{(k+1)}$ by decreasing $\kappa^{(k)}$ by some amount
- Solve for $z^*(\kappa^{(k+1)})$ starting from $z^*(\kappa^{(k)})$
- If method converges, it converges to the optimal solution, i.e., $z^{(k)} \to z^*$ for $\kappa \to 0$

Barrier Interior-point Method

$$\min_z \{f(z) \mid g(z) \le 0\}$$

Input: strictly feasible z, $\kappa := \kappa^{(0)}$, $0 < \mu < 1$, tolerance $\epsilon > 0$

repeat

1. *Centering step:* Compute $z^*(\kappa)$ by minimizing $f(z) + \kappa\phi(z)$ starting from z
2. *Update* $z := z^*(\kappa)$
3. *Stopping criterion:* Stop if $m\kappa < \epsilon$
4. *Decrease barrier parameter:* $\kappa := \mu\kappa$

- Several heuristics for choice of $\kappa^{(0)}$ and other parameters[1]
- Terminates with $f(z^*) - p^* \le \epsilon$
- Steps 1-4 represent one outer iteration
- Step 1: Solve unconstrained minimization problem

[1] More details in Convex Optimization, S. Boyd and L. Vandenberghe

Barrier Interior-point Method

$$\min_z \{f(z) \mid g(z) \leq 0\}$$

Input: strictly feasible z, $\kappa := \kappa^{(0)}$, $0 < \mu < 1$, tolerance $\epsilon > 0$

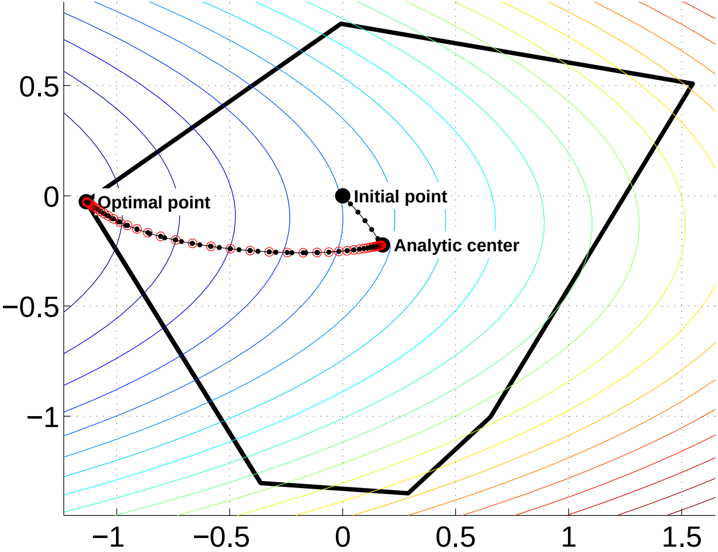
repeat

1. **Centering step: Compute** $z^*(\kappa)$ **by minimizing** $f(z) + \kappa\phi(z)$ **starting from** z
2. *Update* $z := z^*(\kappa)$
3. *Stopping criterion:* Stop if $m\kappa < \epsilon$
4. *Decrease barrier parameter:* $\kappa := \mu\kappa$

- Several heuristics for choice of $\kappa^{(0)}$ and other parameters[1]
- Terminates with $f(z^*) - p^* \leq \epsilon$
- Steps 1-4 represent one outer iteration
- Step 1: Solve unconstrained minimization problem

[1] More details in Convex Optimization, S. Boyd and L. Vandenberghe

Example - Quadratic Program



Outline

1. Optimization in MPC

2. Main Concepts

3. Convex Optimization

4. Linear and Quadratic Programming

5. Constrained Minimization: Interior-point Methods

   - Concept
   - Unconstrained Minimization
   - Barrier Interior-Point Method

6. Summary of Exercise Session

Optimality Conditions for Unconstrained Problems

Consider the unconstrained optimization problem

$$\min_{z} \; f(z) \quad \text{with } f : \mathbb{R}^z \to \mathbb{R}$$

Optimality Conditions for Unconstrained Problems

Consider the unconstrained optimization problem

$$\min_z \ f(z) \quad \text{with } f : \mathbb{R}^z \to \mathbb{R}$$

Theorem: Necessary condition

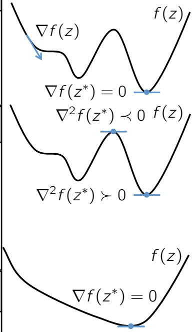
Assume $f(\cdot)$ differentiable at z^*. If z^* is a local minimizer, then $\nabla f(z^*) = 0$.

Theorem: Sufficient condition

Assume that $f(\cdot)$ is twice differentiable at z^*. If $\nabla f(z^*) = 0$ and the Hessian of $f(z)$ at z^* is positive definite, i.e. $\nabla^2 f(z^*) \succ 0$, then z^* is a local minimizer.

Theorem: Necessary and sufficient condition

Assume $f(\cdot)$ differentiable at z^*. If f is convex, then z^* is a global minimizer if and only if $\nabla f(z^*) = 0$.



For more details and proofs see, e.g., M.S. Bazaraa, H.D. Sherali, and C.M. Shetty. Nonlinear Programming Theory and Algorithms. John Wiley & Sons, Inc., New York, 1993.

Unconstrained Minimization

$$\min_z \; f(z) \quad \text{with } f : \mathbb{R}^z \to \mathbb{R}$$

- f convex, twice continuously differentiable
- We assume optimal value $p^* = \min_z f(z)$ is attained (and finite)

Unconstrained minimization methods

- Generate sequence of points $z^{(k)}$ in domain of f with

$$f(z^{(k)}) \to p^* \quad \text{for} \quad k \to \infty$$

- Can be interpreted as iterative methods for solving optimality condition

$$\nabla f(z^*) = 0$$

(nonlinear set of equations, usually no analytical solution)

Descent Methods

$$z^{(k+1)} = z^{(k)} + t^{(k)} \Delta z^{(k)} \quad \text{with } f(z^{(k+1)}) < f(z^{(k)})$$

- Δz is the **step** or **search direction**
- t is the **step size** or **step length**
- $f(z^{(k+1)}) < f(z^{(k)})$, i.e., Δz is a **descent direction**
- There exists a $t > 0$ such that $f(z^{(k+1)}) < f(z^{(k)})$ if $\nabla f(z)^T \Delta z < 0$

Descent Methods

$$z^{(k+1)} = z^{(k)} + t^{(k)}\Delta z^{(k)} \quad \text{with } f(z^{(k+1)}) < f(z^{(k)})$$

- Δz is the **step** or **search direction**
- t is the **step size** or **step length**
- $f(z^{(k+1)}) < f(z^{(k)})$, i.e., Δz is a **descent direction**
- There exists a $t > 0$ such that $f(z^{(k+1)}) < f(z^{(k)})$ if $\nabla f(z)^T \Delta z < 0$

General descent method:

Input: starting point $z \in$ domain of f

repeat

1. Compute a *descent direction* Δz
2. *Line search*: Choose step size $t > 0$ such that $f(z + t\Delta z) < f(z)$
3. *Update* $z := z + t\Delta z$

until stopping criterion is satisfied

Descent Directions

Lots of ways to choose descent directions:

- Gradient descent: $\Delta z := -\nabla f(z)$

$$\nabla f(z)^T \Delta z = -\nabla f(z)^T \nabla f(z) = -\|\nabla f(z)\| < 0$$

  Tends to be too aggressive (zig-zagging).

- Direction as function of several previous gradients:
  - Conjugate gradient method
  - Fast gradient method

  Much better, but still limited

- Best: Newton method

$$\Delta z = -\nabla^2 f(z)^{-1} \nabla f(z)$$

  Cost: Must invert the Hessian

Newton's Method

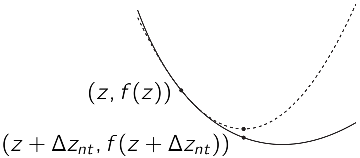
$$\Delta z_{nt} = -\nabla^2 f(z)^{-1} \nabla f(z)$$

- Interpretation: $z + \Delta z_{nt}$ minimizes second order approximation

$$\hat{f}(z+v) = f(z) + \nabla f(z)^T v + \frac{1}{2} v^T \nabla^2 f(z) v$$

Optimality condition: $\nabla \hat{f}(z+v^*) = 0$

$$\nabla f(z) + \nabla^2 f(z) v^* = 0$$
$$\Rightarrow \nabla^2 f(z) v^* = -\nabla f(z)$$



- Decent direction:

$$\nabla f(z)^T \Delta z_{nt} = -\nabla f(z)^T \nabla^2 f(z)^{-1} \nabla f(z) < 0$$

f convex implies that $\nabla^2 f(z) \succeq 0$

- If z is close to optimum, $\|\nabla f(z)\|_2$ converges to zero quadratically (**extremly** quickly)

Line-search

Choose step size $t > 0$ such that $f(z + t\Delta z) < f(z)$

$$t^\star = \text{argmin}_{t>0}\, f(z + t\Delta z)$$

f is convex, and so $f(z + t\Delta z)$ is a single-variable convex function in t.

Options:

- Solve exactly using bisection search
  - Time consuming, requires many evaluations of f
- Solve very approximately using *backtracking search*
  - Much faster, step size very rough
  - Accuracy doesn't usually matter much

Outline

1. Optimization in MPC
2. Main Concepts
3. Convex Optimization
4. Linear and Quadratic Programming
5. Constrained Minimization: Interior-point Methods
   - Concept
   - Unconstrained Minimization
   - Barrier Interior-Point Method
6. Summary of Exercise Session

Barrier Interior-point Method

$$\min_z \{f(z) \mid g(z) \leq 0\}$$

Input: strictly feasible z, $\kappa := \kappa^{(0)}$, $0 < \mu < 1$, tolerance $\epsilon > 0$

repeat

1. *Centering step:* Compute $z^*(\kappa)$ by minimizing $f(z) + \kappa\phi(z)$ starting from z
2. *Update* $z := z^*(\kappa)$
3. *Stopping criterion:* Stop if $m\kappa < \epsilon$
4. *Decrease barrier parameter:* $\kappa := \mu\kappa$

- Several heuristics for choice of $\kappa^{(0)}$ and other parameters[1]
- Terminates with $f(z^*) - p^* \leq \epsilon$
- Steps 1-4 represent one outer iteration
- Step 1: Solve unconstrained minimization problem

[1] More details in Convex Optimization, S. Boyd and L. Vandenberghe

Centering Step using Newton's Method

Centering Step: Compute $z^*(\kappa)$ by solving

$$\min_z f(z) + \kappa\phi(z)$$

Apply algorithm of general descent method:

Input: starting point $z \in$ domain of f
repeat
1. Compute *descent direction* Δz
2. *Line search*: Choose step size $t > 0$ such that $f(z + t\Delta z) < f(z)$
3. *Update* $z := z + t\Delta z$

until stopping criterion is satisfied

with

- Descent direction: Newton direction
- Line search: Adapt to satisfy inequality constraints

Centering Step using Newton's Method

Newton direction:

- Δz_{nt} minimizes second order approximation

$$\begin{aligned}\hat{f}(z+v) =& f(z) + \kappa\phi(z) + \nabla f(z)^T v + \kappa \nabla \phi(z)^T v \\ &+ \frac{1}{2} v^T \nabla^2 f(z) v + \frac{1}{2}\kappa v^T \nabla^2 \phi(z) v\end{aligned}$$

- Newton direction for barrier method is given by solution of

$$(\nabla^2 f(z) + \kappa \nabla^2 \phi(z))\Delta z_{nt} = -\nabla f(z) - \kappa \nabla \phi(z)$$

Line search: Consists of two steps:

- Find $t_{max} = \text{argmax}_{0 \le t \le 1}\{t \mid g_1(z + t\Delta z) < 0, \dots, g_m(z + t\Delta z) < 0\}$
- FInd $t^* = \text{argmin}_{t \ge 0}\{f(z + t\Delta z)\}$

both either solved exactly or through backtracking.

Newton step for Quadratic Programming

$$\min_z \{\frac{1}{2} z^T H z \mid Gz \leq d\}$$

- Barrier method:

$$\min_z f(z) + \kappa \phi(z) = \min_z \frac{1}{2} z^T H z - \kappa \sum_{i=1}^{m} \log(d_i - g_i z)$$

  where $g_1, \ldots, g_m$ are the rows of G.

- The gradient and Hessian of the barrier function are:

$$\nabla \phi(z) = \sum_{i=1}^{m} \frac{1}{d_i - g_i z} g_i^T, \nabla^2 \phi(z) = \sum_{i=1}^{m} \frac{1}{(d_i - g_i z)^2} g_i^T g_i$$

- Newton step:

$$(\nabla^2 f(z) + \kappa \nabla^2 \phi(z)) \Delta z_{nt} = -\nabla f(z) - \kappa \nabla \phi(z)$$

$$(H + \kappa \sum_{i=1}^{m} \frac{1}{(d_i - g_i z)^2} g_i^T g_i) \Delta z_{nt} = -Hz - \kappa \sum_{i=1}^{m} \frac{1}{d_i - g_i z} g_i^T$$

Barrier IPM for Quadratic Programming

$$\min_z \{ \frac{1}{2} z^T H z \mid Gz \le d \}$$

Input: strictly feasible z, $\kappa := \kappa^{(0)}$, $0 < \mu < 1$, tolerance $\epsilon > 0$

repeat

1. *Centering step:*
2. **repeat**

   2.1 Compute *search direction* Δz:

$$(H + \kappa \sum_{i=1}^{m} \frac{1}{(d_i - g_i z)^2} g_i^T g_i) \Delta z = -Hz - \kappa \sum_{i=1}^{m} \frac{1}{d_i - g_i z} g_i^T$$

   2.2 *Line search*: Choose step size $t > 0$

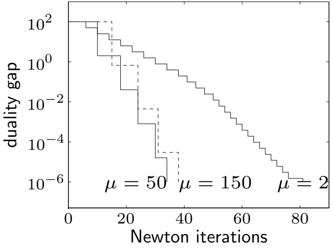
   2.3 *Update* $z := z + t\Delta z$

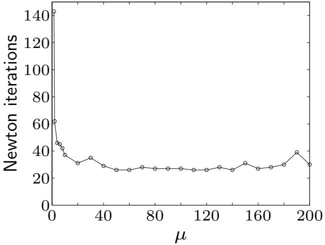
3. **until** stopping criterion is satisfied
4. *Decrease barrier parameter:* $\kappa := \mu\kappa$

until $m\kappa < \epsilon$

Remarks on Barrier Method

- Choice of μ involves trade-off: small μ few outer iterations, but more inner iterations to compute $z^{(k+1)}$ from $z^{(k)}$ (typical values $\mu = 0.1 - 0.05$)
- Good convergence properties for a wide range of parameters μ
  Example: LP with 100 inequalities, 50 variables





(Note that the μ shown here in the plots is $1/\mu$ from the lectures)

- Barrier method requires strictly feasible initial point
  Phase I method, e.g., $\min_{z,s}\{s \mid g(z) \leq s\}$
- Barrier method can be similarly applied to problems with additional equality constraints

Barrier Interior-point Method

$$\min_z \{f(z) \mid g(z) \leq d,\ Cz = b\}$$

Input: strictly feasible z, $\kappa := \kappa^{(0)}$, $\mu > 1$, tolerance $\epsilon > 0$

repeat

1. **Centering step: Compute $z^*(\kappa)$ by minimizing $f(z) + \kappa\phi(z)$ subject to $Cz = b$ starting from z**
2. *Update* $z := z^*(\kappa)$
3. *Stopping criterion:* Stop if $m\kappa < \epsilon$
4. *Decrease barrier parameter:* $\kappa := \mu\kappa$

- Several heuristics for choice of $\kappa^{(0)}$ and other parameters[2]
- Terminates with $f(z^*) - p^* \leq \epsilon$
- Steps 1-4 represent one outer iteration
- Step 1: Solve unconstrained minimization problem

[2] More details in Convex Optimization, S. Boyd and L. Vandenberghe, 2004

Centering Step using Newton's Method

Centering Step: Compute $z^*(\kappa)$ by solving

$$\begin{aligned} \min\ & f(z) + \kappa\phi(z) \\ \text{s.t.}\ & Cz = d \end{aligned}$$

- Newton step Δz_{nt} for minimization with equality constraints is given by solution of

$$\begin{bmatrix} \nabla^2 f(z) + \kappa\nabla^2\phi(z) & C^T \\ C & \end{bmatrix} \begin{bmatrix} \Delta z_{nt} \\ \nu \end{bmatrix} = - \begin{bmatrix} \nabla f(z) + \kappa\nabla\phi(z) \\ 0 \end{bmatrix}$$

- Same interpretation as Newton step for unconstrained problem: $z + \Delta z_{nt}$ minimizes second order approximation

$$\begin{aligned} \min\ & \nabla f(z)^T v + \kappa\nabla\phi(z)^T v + \frac{1}{2} v^T \nabla^2 f(z) v + \frac{1}{2}\kappa v^T \nabla^2\phi(z) v \\ \text{s.t.}\ & Cv = 0 \end{aligned}$$

Recap: Interior-point Methods

Barrier method

- Intuition: Follow central path to the optimal solution
- Log barrier function ensures satisfaction of inequality constraints
- Unconstrained or equality constrained problems can be solved efficiently using Newton's method

'Modern' methods: Primal-dual methods

- Often more efficient than barrier method, can exhibit better than linear convergence
- Cost per iteration same as barrier method
- Allow for infeasible start (w.r.t. both equality and inequality constraints)
- Most efficient in practice: Mehrotra's predictor-corrector method[3]

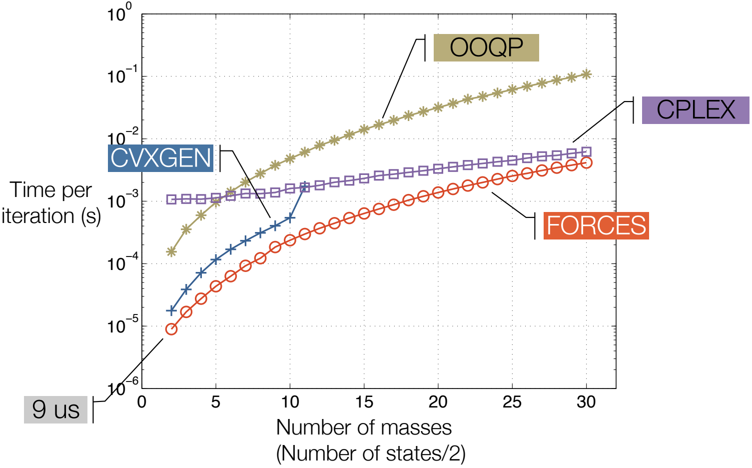
Interior-point methods are very efficient for range of optimization problems, e.g. LPs, QPs, second-order cone programs, semidefinite programs.

---

[3] See, e.g., Numerical Optimization, J. Nocedal and S. Wright, 2006 Springer

How Fast?

Time per iteration for MPC problem on desktop PC.
Total time will be $\sim 10\times$ slower.



[A. Domahidi, A.U. Zgraggen, M.N. Zeilinger, M. Morari and C.N. Jones, 2012]

Outline

Exercise Session #2

Implement the barrier method for the QP

$$
\begin{aligned}
\min_z \quad & \frac{1}{2} z^T H z + q^T z \\
\text{s.t.} \quad & Gz \leq d
\end{aligned}
$$

Download code from moodle

Exercises:

1. Compute the search direction and complete the implementation of the barrier method.

2. Investigate relationship between problem and tuning parameters and convergence rate.